OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4757

Further Applications of Advanced Mathematics (FP3)

Monday **12 JUNE 2006** Afternoon 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 5 printed pages and 3 blank pages.

Option 1: Vectors

1 Four points have coordinates $\mathrm{A}(-2, -3, 2)$, $\mathrm{B}(-3, 1, 5)$, $\mathrm{C}(k, 5, -2)$ and $\mathrm{D}(0, 9, k)$.

(i) Find the vector product $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{CD}}$. [4]

(ii) For the case when AB is parallel to CD,

(*A*) state the value of k, [1]

(*B*) find the shortest distance between the parallel lines AB and CD, [6]

(*C*) find, in the form $ax + by + cz + d = 0$, the equation of the plane containing AB and CD. [3]

(iii) When AB is not parallel to CD, find the shortest distance between the lines AB and CD, in terms of k. [4]

(iv) Find the value of k for which the line AB intersects the line CD, and find the coordinates of the point of intersection in this case. [6]

Option 2: Multi-variable calculus

2 A surface has equation $x^2 - 4xy + 3y^2 - 2z^2 - 63 = 0$.

(i) Find a normal vector at the point (x, y, z) on the surface. [4]

(ii) Find the equation of the tangent plane to the surface at the point $\mathrm{Q}(17, 4, 1)$. [4]

(iii) The point $(17 + h, 4 + p, 1 - h)$, where h and p are small, is on the surface and is close to Q.

Find an approximate expression for p in terms of h. [4]

(iv) Show that there is no point on the surface where the normal line is parallel to the z-axis. [4]

(v) Find the two values of k for which $5x - 6y + 2z = k$ is a tangent plane to the surface. [8]

Option 3: Differential geometry

3 The curve C has parametric equations $x = 2t^3 - 6t$, $y = 6t^2$.

(i) Find the length of the arc of C for which $0 \leqslant t \leqslant 1$. [6]

(ii) Find the area of the surface generated when the arc of C for which $0 \leqslant t \leqslant 1$ is rotated through 2π radians about the x-axis. [5]

(iii) Show that the equation of the normal to C at the point with parameter t is

$$y = \frac{1}{2}\left(\frac{1}{t} - t\right)x + 2t^2 + t^4 + 3.$$ [4]

(iv) Find the cartesian equation of the envelope of the normals to C. [6]

(v) The point $P(64, a)$ is the centre of curvature corresponding to a point on C. Find a. [3]

Option 4: Groups

4 The group G consists of the 8 complex matrices $\{\mathbf{I}, \mathbf{J}, \mathbf{K}, \mathbf{L}, -\mathbf{I}, -\mathbf{J}, -\mathbf{K}, -\mathbf{L}\}$ under matrix multiplication, where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \mathrm{j} & 0 \\ 0 & -\mathrm{j} \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & \mathrm{j} \\ \mathrm{j} & 0 \end{pmatrix}.$$

(i) Copy and complete the following composition table for G. [6]

	I	**J**	**K**	**L**	**–I**	**–J**	**–K**	**–L**
I	**I**	**J**	**K**	**L**	**–I**	**–J**	**–K**	**–L**
J	**J**	**–I**	**L**	**–K**	**–J**	**I**	**–L**	**K**
K	**K**	**–L**	**–I**					
L	**L**	**K**						
–I	**–I**	**–J**						
–J	**–J**	**I**						
–K	**–K**	**L**						
–L	**–L**	**–K**						

(Note that $\mathbf{JK} = \mathbf{L}$ and $\mathbf{KJ} = -\mathbf{L}$.)

(ii) State the inverse of each element of G. [3]

(iii) Find the order of each element of G. [3]

(iv) Explain why, if G has a subgroup of order 4, that subgroup must be cyclic. [4]

(v) Find all the proper subgroups of G. [5]

(vi) Show that G is not isomorphic to the group of symmetries of a square. [3]

Option 5: Markov chains

5 A local hockey league has three divisions. Each team in the league plays in a division for a year. In the following year a team might play in the same division again, or it might move up or down one division.

This question is about the progress of one particular team in the league. In 2007 this team will be playing in either Division 1 or Division 2. Because of its present position, the probability that it will be playing in Division 1 is 0.6, and the probability that it will be playing in Division 2 is 0.4.

The following transition probabilities apply to this team from 2007 onwards.

- When the team is playing in Division 1, the probability that it will play in Division 2 in the following year is 0.2.
- When the team is playing in Division 2, the probability that it will play in Division 1 in the following year is 0.1, and the probability that it will play in Division 3 in the following year is 0.3.
- When the team is playing in Division 3, the probability that it will play in Division 2 in the following year is 0.15.

This process is modelled as a Markov chain with three states corresponding to the three divisions.

(i) Write down the transition matrix. [3]

(ii) Determine in which division the team is most likely to be playing in 2014. [6]

(iii) Find the equilibrium probabilities for each division for this team. [3]

In 2015 the rules of the league are changed. A team playing in Division 3 might now be dropped from the league in the following year. Once dropped, a team does not play in the league again.

- The transition probabilities from Divisions 1 and 2 remain the same as before.
- When the team is playing in Division 3, the probability that it will play in Division 2 in the following year is 0.15, and the probability that it will be dropped from the league is 0.1.

The team plays in Division 2 in 2015.

The new situation is modelled as a Markov chain with four states: ‘Division1’, ‘Division 2’, ‘Division 3’ and ‘Out of league’.

(iv) Write down the transition matrix which applies from 2015. [3]

(v) Find the probability that the team is still playing in the league in 2020. [5]

(vi) Find the first year for which the probability that the team is out of the league is greater than 0.5. [4]

Mark Scheme 4757
June 2006

1 (i)	$\begin{pmatrix}-1\\4\\3\end{pmatrix}\times\begin{pmatrix}-k\\4\\k+2\end{pmatrix}=\begin{pmatrix}4k-4\\2-2k\\4k-4\end{pmatrix}$ $\left[\ =2(k-1)\begin{pmatrix}2\\-1\\2\end{pmatrix}\ \right]$	B1 M1 A2 **4**	$\overrightarrow{AB}$ and $\overrightarrow{CD}$ *(Condone* $\overrightarrow{BA}$ *and* $\overrightarrow{DC}$ *)* Evaluating vector product Give A1 ft for one element correct
(ii)(*A*)	$k=1$	B1 **1**	
(*B*)	$\overrightarrow{CA}\times\overrightarrow{AB}=\begin{pmatrix}-3\\-8\\4\end{pmatrix}\times\begin{pmatrix}-1\\4\\3\end{pmatrix}=\begin{pmatrix}-40\\5\\-20\end{pmatrix}$ Distance is $\dfrac{\lvert\overrightarrow{CA}\times\overrightarrow{AB}\rvert}{\lvert\overrightarrow{AB}\rvert}$ $=\dfrac{45}{\sqrt{26}}$ (≈ 8.825)	M1 M1 A1 M1 M1 A1 **6**	For appropriate vector product Evaluation *Dependent on previous M1* Method for finding shortest distance *Dependent on <u>first</u> M1* Calculating magnitudes *Dependent on previous M1* Accept 8.82 to 8.83
	OR $\overrightarrow{CP}\,.\,\overrightarrow{AB}=\begin{pmatrix}-2-\lambda-1\\-3+4\lambda-5\\2+3\lambda+2\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\3\end{pmatrix}=0$ M2A1 $\overrightarrow{CP}=\dfrac{1}{26}\begin{pmatrix}-95\\-140\\155\end{pmatrix}$ Distance is $\dfrac{\sqrt{52650}}{26}$ M1 M1A1		Finding $\overrightarrow{CP}$ *Dependent on previous M1* *Dependent on previous M1*
(*C*)	Normal vector is $\overrightarrow{CA}\times\overrightarrow{AB}=\begin{pmatrix}-40\\5\\-20\end{pmatrix}=-5\begin{pmatrix}8\\-1\\4\end{pmatrix}$ Equation of plane is $8x-y+4z=-16+3+8$ $8x-y+4z+5=0$	M1 M1 A1 **3**	 *Dependent on previous M1* Allow $-40x+5y-20z=25$ etc
(iii)	$\dfrac{\overrightarrow{AC}\,.\,(\overrightarrow{AB}\times\overrightarrow{CD})}{\lvert\overrightarrow{AB}\times\overrightarrow{CD}\rvert}=\dfrac{\begin{pmatrix}k+2\\8\\-4\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\2\end{pmatrix}(2k-2)}{3(2k-2)}$ Shortest distance is $\left\lvert\dfrac{2k-12}{3}\right\rvert$	M1 M1 A1 ft A1 **4**	For $\overrightarrow{AC}\,.\,(\overrightarrow{AB}\times\overrightarrow{CD})$ Fully correct method *(evaluation not required) Dependent on previous M1* Correct evaluated expression for distance ft from (i) Simplified answer *Modulus not required*

(iv)	Intersect when $k=6$ $-2-\lambda=6-6\mu$ $-3+4\lambda=5+4\mu$ $2+3\lambda=-2+8\mu$ Solving, $\lambda=4,\ \mu=2$ Point of intersection is $(-6,\ 13,\ 14)$	B1 ft M1 A1 ft M1 A1 A1 **6**	 Forming at least two equations Two correct equations Solving to obtain λ or μ *Dependent on previous M1* One value correct
	$-2-\lambda=k-k\mu$ OR $-3+4\lambda=5+4\mu$ M1 $2+3\lambda=-2+(k+2)\mu$ A1 Solving, $k=6$ M1A1 $\lambda=4,\ \mu=2$ A1 Point of intersection is $(-6,\ 13,\ 14)$ A1		Forming three equations All equations correct *Dependent on previous M1* One value correct

2 (i)	Normal vector is $\begin{pmatrix} 2x-4y \\ -4x+6y \\ -4z \end{pmatrix}$	M1 A1 A1 A1 **4**	Partial differentiation Condone $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \lambda \begin{pmatrix} 2x-4y \\ -4x+6y \\ -4z \end{pmatrix}$ *For 4 marks the normal must appear as a vector (isw)*
(ii)	At Q normal vector is $\begin{pmatrix} 18 \\ -44 \\ -4 \end{pmatrix}$ Tangent plane is $18x - 44y - 4z = 306 - 176 - 4 = 126$ $9x - 22y - 2z = 63$	M1 M1 M1 A1 **4**	 For $18x - 44y - 4z$ *Dependent on previous M1* Using Q to find constant Accept any correct form
(iii)	$18\delta x - 44\delta y - 4\delta z \approx 0$ $18h - 44p - 4(-h) \approx 0$ $p \approx \frac{1}{2}h$	M1 A1 ft M1 A1 **4**	For $18\delta x - 44\delta y - 4\delta z$ *If left in terms of x, y, z: M1A0M1A0*
	OR $9(17+h) - 22(4+p) - 2(1-h) \approx 63$ M2A1 ft $p \approx \frac{1}{2}h$ A1		
	OR $(17+h)^2 - 4(17+h)(4+p) + ... = 0$ $-44p + 22h \approx 0$ M2A1 $p \approx \frac{1}{2}h$ A1		Neglecting second order terms
	OR $p = \dfrac{4h + 44 \pm \sqrt{28h^2 + 88h + 1936}}{6}$ M2A1 $p \approx \frac{1}{2}h$ A1		
(iv)	Normal parallel to *z*-axis requires $2x - 4y = 0$ and $-4x + 6y = 0$ $x = y = 0$; then $-2z^2 - 63 = 0$ No solutions; hence no such points	M1A1 ft M1 A1 (ag) **4**	 Correctly shown
	OR $2x - 4y = -4x + 6y$, so $y = \frac{3}{5}x$ $-\frac{8}{25}x^2 - 2z^2 - 63 = 0$, hence no points M2A2		Similarly if only $2x - 4y = 0$ used
(v)	$2x - 4y = 5\lambda$ $-4x + 6y = -6\lambda$ $-4z = 2\lambda$ $x = -\frac{3}{2}\lambda$, $y = -2\lambda$, $z = -\frac{1}{2}\lambda$ Substituting into equation of surface $\frac{9}{4}\lambda^2 - 12\lambda^2 + 12\lambda^2 - \frac{1}{2}\lambda^2 - 63 = 0$ $\lambda = \pm 6$	M1A1 ft M1 M1 M1 M1	 Obtaining *x, y, z* in terms of λ or $x = 3z$, $y = 4z$ Obtaining a value of λ (or equivalent)

	Point $(-9,\ -12,\ -3)$ gives $k=-45+72-6=21$ Point $(9,\ 12,\ 3)$ gives $k=45-72+6=-21$	A1 A1 **8**	Using a point to find *k* *If* $\lambda=1$ *is assumed:* *M0M1M0M0M1*

3 (i)	$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2=(6t^2-6)^2+(12t)^2$ $=36t^4+72t^2+36$ $=36(t^2+1)^2$ Arc length is $\int_0^1 6(t^2+1)\,\mathrm{d}t$ $=\left[\ 2t^3+6t\ \right]_0^1$ $=8$	M1A1 A1 M1 A1 A1 **6**	 Using $\int\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ For $2t^3+6t$
(ii)	Curved surface area is $\int 2\pi\,y\,\mathrm{d}s=\int_0^1 2\pi(6t^2)6(t^2+1)\,\mathrm{d}t$ $=\pi\left[\ \tfrac{72}{5}t^5+24t^3\ \right]_0^1$ $=\frac{192\pi}{5}\quad(\approx 120.6)$	M1 A1 M1 A1 A1 **5**	Using $\int ...y\,\mathrm{d}s$ (in terms of t) with 'ds' the same as in (i) Any correct integral form in terms of t *(limits required)* Integration For $\pi(\tfrac{72}{5}t^5+24t^3)$
(iii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{12t}{6t^2-6}\quad\left(=\frac{2t}{t^2-1}\right)$ Equation of normal is $y-6t^2=\frac{1-t^2}{2t}(x-2t^3+6t)$ $y-6t^2=\frac{1}{2}\left(\frac{1}{t}-t\right)x-t^2(1-t^2)+3(1-t^2)$ $y=\frac{1}{2}\left(\frac{1}{t}-t\right)x+2t^2+t^4+3$	M1 A1 M1 A1 (ag) **4**	Method of differentiation *At least one intermediate step required* Correctly obtained
(iv)	Differentiating partially with respect to t $0=\frac{1}{2}\left(-\frac{1}{t^2}-1\right)x+4t+4t^3$ $\frac{1}{2t^2}(1+t^2)x=4t(1+t^2)$ $x=8t^3$ $t=\tfrac{1}{2}x^{\frac{1}{3}}$, so $y=\tfrac{1}{2}(2x^{-\frac{1}{3}}-\tfrac{1}{2}x^{\frac{1}{3}})x+\tfrac{1}{2}x^{\frac{2}{3}}+\tfrac{1}{16}x^{\frac{4}{3}}+3$ $y=\tfrac{3}{2}x^{\frac{2}{3}}-\tfrac{3}{16}x^{\frac{4}{3}}+3$	M1 A2 M1 M1 A1 **6**	 Give A1 if just one error or omission For obtaining $a\,x=b\,t^3$ Eliminating t

(v)	P lies on the envelope of the normals Hence $a = \frac{3}{2} \times 64^{\frac{2}{3}} - \frac{3}{16} \times 64^{\frac{4}{3}} + 3$ $= -21$	M1 M1 A1 **3**	Or a fully correct method for finding the centre of curvature at a general pt [$(8t^3,\ 6t^2 - 3t^4 + 3)$] Or $t = 2$ and $a = 6 \times 2^2 - 3 \times 2^4 + 3$

4 (i)

	I	J	K	L	–I	–J	–K	–L
I	I	J	K	L	–I	–J	–K	–L
J	J	–I	L	–K	–J	I	–L	K
K	K	–L	–I	**J**	**–K**	**L**	**I**	**–J**
L	L	K	**–J**	**–I**	**–L**	**–K**	**J**	**I**
–I	–I	–J	**–K**	**–L**	**I**	**J**	**K**	**L**
–J	–J	I	**–L**	**K**	**J**	**–I**	**L**	**–K**
–K	–K	L	**I**	**–J**	**K**	**–L**	**–I**	**J**
–L	–L	–K	**J**	**I**	**L**	**K**	**–J**	**–I**

B6 **6**

Give B5 for 30 (bold) entries correct
Give B4 for 24 (bold) entries correct
Give B3 for 18 (bold) entries correct
Give B2 for 12 (bold) entries correct
Give B1 for 6 (bold) entries correct

(ii)

Element	**I**	**J**	**K**	**L**	**–I**	**–J**	**–K**	**–L**
Inverse	**I**	**–J**	**–K**	**–L**	**–I**	**J**	**K**	**L**

B3 **3**

Give B2 for six correct
Give B1 for three correct

(iii)

Element	**I**	**J**	**K**	**L**	**–I**	**–J**	**–K**	**–L**
Order	1	4	4	4	2	4	4	4

B3 **3**

Give B2 for six correct
Give B1 for three correct

(iv)

Only two elements of *G* do not have order 4; so any subgroup of order 4 must contain an element of order 4	M1A1	
A subgroup of order 4 is cyclic if it contains an element of order 4	B1	(may be implied)
Hence any subgroup of order 4 is cyclic	A1 **4**	For completion

OR If a group of order 4 is not cyclic, it contains three elements of order 2 B1
G has only one element of order 2; so this cannot occur M1A1
So any subgroup of order 4 is cyclic A1

(v)

$\{\mathbf{I},\ -\mathbf{I}\}$	B1	
$\{\mathbf{I},\ \mathbf{J},\ -\mathbf{I},\ -\mathbf{J}\}$	B1	
$\{\mathbf{I},\ \mathbf{K},\ -\mathbf{I},\ -\mathbf{K}\}$	B1	
$\{\mathbf{I},\ \mathbf{L},\ -\mathbf{I},\ -\mathbf{L}\}$	B1	
	B1 **5**	For $\{\mathbf{I},\ -\mathbf{I}\}$, at least one correct subgroup of order 4, and no wrong subgroups. This mark is lost if *G* or $\{\mathbf{I}\}$ is included

(vi)	The symmetry group has 5 elements of order 2 (4 reflections and rotation through $180°$) *G* has only one element of order 2, hence *G* is not isomorphic to the symmetry group	M1 A1 A1 **3**	Considering elements of order 2 (or self-inverse elements) Identification of at least two elements of order 2 in the symmetry group For completion

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0.8 & 0.1 & 0\\ 0.2 & 0.6 & 0.15\\ 0 & 0.3 & 0.85\end{pmatrix}$	B1B1B1 **3**	For the three columns
(ii)	$\mathbf{P}^7\begin{pmatrix}0.6\\0.4\\0\end{pmatrix}=\begin{pmatrix}0.3204 & 0.1545 & 0.0927\\ 0.3089 & 0.2895 & 0.2780\\ 0.3706 & 0.5560 & 0.6293\end{pmatrix}\begin{pmatrix}0.6\\0.4\\0\end{pmatrix}=\begin{pmatrix}0.254\\0.301\\0.445\end{pmatrix}$ Division 3 is the most likely	M1 M1 A1 M1 A1 A1 **6**	Considering $\mathbf{P}^7$ (or $\mathbf{P}^8$ or $\mathbf{P}^6$) Evaluating a power of **P** For $\mathbf{P}^7$ (Allow ± 0.001 throughout) Evaluation of probabilities One probability correct Correctly determined
(iii)	$\mathbf{P}^n\to\begin{pmatrix}0.1429 & 0.1429 & 0.1429\\ 0.2857 & 0.2857 & 0.2857\\ 0.5714 & 0.5714 & 0.5714\end{pmatrix}$ Equilibrium probabilities are 0.143, 0.286, 0.571	M1 M1 A1 **3**	Considering powers of **P** Obtaining limit *Must be accurate to 3 dp if given as decimals*
	OR $\mathbf{P}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}p\\q\\r\end{pmatrix}\Rightarrow \begin{matrix}0.8p+0.1q=p\\ 0.2p+0.6q+0.15r=q\\ 0.3q+0.85r=r\end{matrix}$ M1 $q=2p,\ r=2q=4p$ and $p+q+r=1$ M1 $p=\frac{1}{7},\ q=\frac{2}{7},\ r=\frac{4}{7}$ A1		Obtaining at least two equations Solving (must use $p+q+r=1$)
(iv)	$\mathbf{Q}=\begin{pmatrix}0.8 & 0.1 & 0 & 0\\ 0.2 & 0.6 & 0.15 & 0\\ 0 & 0.3 & 0.75 & 0\\ 0 & 0 & 0.1 & 1\end{pmatrix}$	B1 B1 B1 **3**	Third column Fourth column Fully correct
(v)	$\mathbf{Q}^5\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0.4122 & 0.1566 & 0.0592 & 0\\ 0.3131 & 0.2767 & 0.2052 & 0\\ 0.2369 & 0.4105 & 0.4030 & 0\\ 0.0378 & 0.1563 & 0.3326 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $=\begin{pmatrix}0.1566\\0.2767\\0.4105\\0.1563\end{pmatrix}$ P(still in league)$=1-0.1563$ $=0.844$	M1 M1 A1 M1 A1 ft **5**	Considering $\mathbf{Q}^5$ (or $\mathbf{Q}^6$ or $\mathbf{Q}^4$) Evaluating a power of **Q** For 0.1563 (Allow 0.156 ± 0.001) For $1-a_{4,2}$ ft dependent on M1M1M1
(vi)	P(out of league) is element $a_{4,2}$ in $\mathbf{Q}^n$ When $n=15$, $a_{4,2}=0.4849$ When $n=16$, $a_{4,2}=0.5094$ First year is 2031	M1 M1 A1 A1 **4**	Considering $\mathbf{Q}^n$ for at least two more values of *n* Considering $a_{4,2}$ *Dep on previous M1* For $n=16$ *SR* With no working, $n=16$ stated B3 2031 stated B4

Post-multiplication by transition matrix

5 (i)	$\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0.15 & 0.85 \end{pmatrix}$	B1B1B1 **3**	For the three rows
(ii)	$\begin{pmatrix} 0.6 & 0.4 & 0 \end{pmatrix}\mathbf{P}^7$ $= \begin{pmatrix} 0.6 & 0.4 & 0 \end{pmatrix} \begin{pmatrix} 0.3204 & 0.3089 & 0.3706 \\ 0.1545 & 0.2895 & 0.5560 \\ 0.0927 & 0.2780 & 0.6293 \end{pmatrix}$ $= \begin{pmatrix} 0.254 & 0.301 & 0.445 \end{pmatrix}$ Division 3 is the most likely	M1 M1 A1 M1 A1 A1 **6**	Considering $\mathbf{P}^7$ (or $\mathbf{P}^8$ or $\mathbf{P}^6$) Evaluating a power of **P** For $\mathbf{P}^7$ (Allow ± 0.001 throughout) Evaluation of probabilities One probability correct Correctly determined
(iii)	$\mathbf{P}^n \to \begin{pmatrix} 0.1429 & 0.2857 & 0.5714 \\ 0.1429 & 0.2857 & 0.5714 \\ 0.1429 & 0.2857 & 0.5714 \end{pmatrix}$ Equilibrium probabilities are 0.143, 0.286, 0.571	M1 M1 A1 **3**	Considering powers of **P** Obtaining limit *Must be accurate to 3 dp if given as decimals*
	OR $\begin{pmatrix} p & q & r \end{pmatrix}\mathbf{P} = \begin{pmatrix} p & q & r \end{pmatrix}$ $0.8p + 0.1q = p$ $0.2p + 0.6q + 0.15r = q$ M1 $0.3q + 0.85r = r$ $q = 2p$, $r = 2q = 4p$ and $p + q + r = 1$ M1 $p = \frac{1}{7}$, $q = \frac{2}{7}$, $r = \frac{4}{7}$ A1		Obtaining at least two equations Solving (must use $p + q + r = 1$)
(iv)	$\mathbf{Q} = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.1 & 0.6 & 0.3 & 0 \\ 0 & 0.15 & 0.75 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	B1 B1 B1 **3**	Third row Fourth row Fully correct
(v)	$\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}\mathbf{Q}^5$ $= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.4122 & 0.3131 & 0.2369 & 0.0378 \\ 0.1566 & 0.2767 & 0.4105 & 0.1563 \\ 0.0592 & 0.2052 & 0.4030 & 0.3326 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 0.1566 & 0.2767 & 0.4105 & 0.1563 \end{pmatrix}$ P(still in league) $= 1 - 0.1563$ $= 0.844$	M1 M1 A1 M1 A1 ft **5**	Considering $\mathbf{Q}^5$ (or $\mathbf{Q}^6$ or $\mathbf{Q}^4$) Evaluating a power of **Q** For 0.1563 (Allow 0.156 ± 0.001) For $1 - a_{2,4}$ ft dependent on M1M1M1

(vi)	P(out of league) is element $a_{2,4}$ in $\mathbf{Q}^n$ When $n=15$, $a_{2,4}=0.4849$ When $n=16$, $a_{2,4}=0.5094$ First year is 2031	M1 M1 A1 A1 **4**	Considering $\mathbf{Q}^n$ for at least two more values of *n* Considering $a_{2,4}$ *Dep on previous M1* For $n=16$ *SR* With no working, $n=16$ stated B3 2031 stated B4

4757 - Further Applications of Advanced Mathematics (FP3)

General Comments

There were some excellent scripts, with about 15% of candidates scoring more than 60 marks (out of 72), and a wide range of performance; about 20% of the candidates scored less than 30 marks. Questions 1 and 2 were the most popular, and questions 3 and 5 were the least popular. Some candidates indicated that they were running out of time, and very few presented answers to more than the required three questions. The average marks for the questions were about 14 (out of 24) for questions 1, 2 and 3; about 16 for question 4 and about 18 for question 5.

Comments on Individual Questions

1) **Vectors**

The techniques required in this question were generally well known, although weaker candidates often had difficulty selecting which vectors to use in the formulae, for example using one of the position vectors when a displacement vector such as $\overrightarrow{\mathrm{CD}}$ was required.
Parts (i) and (ii)(*A*) were almost always answered correctly.
Part (ii)(*B*) was often answered correctly, although many candidates tried to use the formula for the distance between skew lines.
Part (ii)(*C*) was answered well.
In part (iii) some candidates continued to take $k = 1$, but the method was very well understood. Many had problems simplifying the answer, especially when the factor $(k - 1)$ had not been taken out of the direction vector for the common perpendicular.
Part (iv) was well answered. The value of *k* could be deduced from part (iii), but this was not always easy when the answer to (iii) was wrong or unsimplified. Even so, it was not difficult to start afresh and find the point of intersection, and *k*, from the three component equations, and many candidates did this successfully.

2) **Multi-variable calculus**

Part (i) was almost always answered correctly. Many candidates gave the equation of the normal line as their final answer, instead of the normal vector, but this was not penalised.
Part (ii) was usually answered correctly, and part (iii) was also well answered, although δz was often taken to be *h* instead of $-h$.
In part (iv) the condition for the normal line to be parallel to the *z*-axis was usually stated correctly, although some thought that it was $\frac{\partial g}{\partial z} = 0$. Quite a number asserted that the simultaneous equations $2x - 4y = 0$ and $-4x + 6y = 0$ were inconsistent, when they are clearly satisfied by $x = y = 0$. Very many candidates completed this part correctly.
There were few correct solutions to part (v). Most candidates wrote $2x - 4y = 5$, $-4x + 6y = -6$, $-4z = 2$ instead of $2x - 4y = 5\lambda$, $-4x + 6y = -6\lambda$, $-4z = 2\lambda$, and these could score only 2 out of the 8 marks.

3) **Differential geometry**

Parts (i), (ii) and (iii) were very often answered correctly.
In part (iv) many candidates differentiated correctly, but further progress depended on dividing by $(1 + t^2)$ to obtain $x = 8t^3$. Very few obtained the correct answer.
In part (v) candidates generally knew that they should substitute $x = 64$ into the answer to part (iv). A few tried to find the centre of curvature at a general point, seldom successfully.

4) **Groups**

In part (i) the completion of the table was generally well done. It was often completely correct, and the most common error was to mix up **J** and $-\mathbf{J}$.

Parts (ii) and (iii) were well answered.

In part (iv) about half the candidates gave a satisfactory explanation. They were expected to say that any set of four elements must include (at least two) elements of order 4, and that a subgroup of order 4 containing an element of order 4 must be cyclic.

In part (v) the four proper subgroups were usually given, but {**I**} was very often given as well; this resulted in the loss of one mark.

In part (vi) candidates usually referred to elements of order 2, but did not always clearly identify at least two such elements in the symmetry group.

5) **Markov chains**

Most of the candidates who chose this question showed a very good understanding of the topic and were able to use their calculators to manipulate the matrices efficiently. No part of the question caused particular difficulty, and about one third of the attempts scored full marks.

In part (iii) the equilibrium probabilities could be found either by solving simultaneous equations or by considering a high power of the transition matrix; the first of these approaches was slightly more commonly used.